7-LOCATED LOCALLY 5-LARGE COMPLEXES ARE ASPHERICAL

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Abstract. We prove that 7-located locally 5-large simplicial complexes are aspherical.

1. Introduction

A simplicial complex is *flag* if every set of vertices pairwise connected by edges spans a simplex. For $k \geq 5$, a flag simplicial complex is *k-large* if it has no induced cycles of length $4 \leq n < k$. A simplicial complex is *locally k-large*, if each of its vertex links is k-large. This notion was introduced by Januszkiewicz and Świątkowski [6], and independently by Haglund [3], as a simplicial analogue of a locally CAT(0) (i.e. nonpositively-curved) cube complex. They showed that such complexes are ubiquitous in any dimension, and come with interesting automorphism groups. A cornerstone feature is that for $k \geq 6$ they are aspherical. The 1-skeleta of simply connected locally 6-large simplicial complexes were studied earlier in graph theory under the name of bridged graphs, see [1] for a survey.

The boundary of the icosahedron is locally 5-large, so in order to obtain asphericity under this weaker condition, Osajda introduced an extra hypothesis of m-location [9] (we will give the definition in a moment). 7-located locally 5-large simplicial complexes include many 3-manifolds, as well as all locally weakly systolic complexes [4], which were studied earlier in [2, 8]. The properties of m-located complexes were investigated in [4, 9].

Maybe the most prominent example of a 7-located locally 5-large simplicial complex is the triangulation of the hyperbolic space $\mathbf{H}^4$ where each of the vertex links is isomorphic to the boundary of the 600-cell. The symmetry group of that triangulation is the Coxeter group with Coxeter diagram the linear graph of length 4 with consecutive labels 5333. We are interested in this triangulation since the associated Artin group is one of the smallest Artin groups for which the $K(\pi, 1)$ conjecture, asking for the contractibility of the associated Artin complex, is still open.

In this paper, we prove the following related result.

Main Theorem. *Every 7-located locally 5-large simplicial complex is aspherical.*

2. Location

Let X be a flag simplicial complex.

Definition 2.1. A *k-wheel* W in X is an induced subcomplex isomorphic to the cone over the k-cycle. We write $W = (v_0, v_1v_2 \cdots v_k)$, where the *centre* v_0 is the cone vertex and $v_1, \ldots, v_k$ are the consecutive vertices of the *boundary* cycle.

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A pair $W = (W_1, W_2)$ of wheels, with $W_1 = (v_0, v_1 \cdots v_k), W_2 = (w_0, w_1 \cdots w_\ell)$, is a (k, ℓ)-*dwheel* if

- $v_k = w_0$,
- $w_\ell = v_0$,
- $v_{k-1} = w_{\ell-1}$, and
- either v_1 equals w_1 or is a neighbour of w_1.

The *boundary* ∂W of the dwheel W is the cycle $v_1 \cdots v_{k-1} w_{\ell-2} \cdots w_1$. (If $w_1 = v_1$, then we discard the redundant w_1.) If $v_1 = w_1$, then we say that W is *planar*.

Definition 2.2. Let $m \geq 4$. X is *m-located*, if for every dwheel $W = (W_1, W_2)$ with $|\partial W| \leq m$, all the vertices of $W_1 \cup W_2$ have a common neighbour in X.

Example 2.3. Let X be the simplicial complex that is the triangulation of the hyperbolic space $\mathbf{H}^4$ where each of the vertex links is isomorphic to the boundary $\mathbf{S}^3_{600}$ of the 600-cell, which is 5-large. Note that the vertex links of $\mathbf{S}^3_{600}$ are isomorphic to the boundary $\mathbf{S}^2_{20}$ of the icosahedron. Since each induced 5-cycle in $\mathbf{S}^3_{600}$ and $\mathbf{S}^2_{20}$ is the boundary of a 5-wheel, each 5-wheel in X can be extended to the join of the 5-cycle and a triangle Δ. Furthermore, each 6-wheel in X can be extended to the join of the 6-cycle and an edge e. Hence X does not contain a planar $(5, 6)$-dwheel (W_1, W_2) with $W_1 \cup W_2$ without a common neighbour, since otherwise appropriate Δ and e are disjoint and so Δ, e, and $v_1 = w_1$ span a simplex of dimension 5 in X, which is a contradiction. The $(5, 5)$-dwheels are excluded similarly, which implies that X is 7-located.

2.1. **Disc diagrams.** A *disc diagram* D is a simplicial complex homeomorphic to a disc. A *disc diagram in* X is a simplicial map $f \colon D \to X$ that is *nondegenerate*, i.e. does not send any edge to a vertex. We say that f has *boundary cycle* $f(\partial D)$. A disc diagram $f \colon D \to X$ is *minimal* if it has minimal area (i.e. the number of triangles in D) among all the diagrams in X with the same boundary cycle. We say that f is *reduced* if it is locally injective at $D \setminus D^0$. The following is a well-known variation of a result by Van Kampen.

Lemma 2.4 ([7, Lem 2.16 and 2.17])**.** *Any homotopically trivial cycle embedded in X^1 is the boundary cycle of a disc diagram in X. Any minimal disc diagram is reduced.*

Lemma 2.5 ([4, Thm B])**.** *If X is 7-located and locally 5-large, then so is D for each minimal disc diagram $D \to X$. In other words, D has no*

- *interior vertices of valence 3 or 4, or*
- *neighbouring interior vertices with valences 5 and 5 or 6.*

Since by (the proof of) [4, Cor 4.7] each D above with $|\partial D| = 4$ has at most five triangles, we have:

Corollary 2.6. *Each D as in Lemma 2.5 is 5-large.*

Remark 2.7. The κ' method from the proof of Proposition 3.2 can be used to give an alternative proof of Corollary 2.6, since in a minimal counterexample to Corollary 2.6 each valence 5 interior vertex has non-positive κ'.

3. Lunar diagrams

In this section, we assume that all disc diagrams D are 7-located and locally 5-large. On the 1-skeleton X^1 of X we consider the path metric d, where all the edges have length 1.

Definition 3.1. Let x and v be distinct vertices of a simplicial complex X, and suppose that γ_1 and γ_2 are geodesics from x to v in X^1 that are disjoint except at the endpoints. A minimal diagram $D \to X$ with boundary $\gamma_1 \cup \gamma_2$ is a *lunar disc diagram between x and v.*

If the identity map $D \to D$ is lunar (that is, if ∂D is a union of geodesics γ_1, γ_2 in D^1 from x to v), then D is *lunar.* Then for a vertex u of ∂D, an interior vertex of D of valence 5 is *u-exposed*, if it is a neighbour of both neighbours of u in ∂D.

By Corollary 2.6, for each u there is at most one u-exposed vertex.

Proposition 3.2. *Let D be a lunar disc diagram between x and v and let v_1, v_2 be the neighbours of v in ∂D. Then*

(i) v_1 and v_2 are neighbours and have a common neighbour closer to x than v_1, v_2, or

(ii) there is a v-exposed neighbour v' of v whose neighbours v_1', v_2' distinct from v, v_1, v_2 are closer to x than v'.

In the proof, we need the following.

Lemma 3.3. *Let D be a lunar disc diagram between x and v and let v_1, v_2 be the neighbours of v in ∂D. Then there is a lunar disc diagram $D' \subseteq D$ between z and v such that*

(1) the path $v_1 v v_2$ lies in $\partial D'$,

(2) the function $d(\cdot, x) - d(\cdot, z)$ is constant on all the vertices of D' at distance ≤ 2 from v,

(3) each vertex on $\partial D'$ has valence at least 4, except possibly for z, v, v_1, or v_2,

(4) each interior vertex of D' of valence 5 that is a neighbour of ≥ 3 vertices of $\partial D'$ is u-exposed with $u \in \{z, v, v_1, v_2\}$.

Remark 3.4. By (1) and (2), if D' satisfies Proposition 3.2(i) or (ii), then so does D.

Proof of Lemma 3.3. Let $D' \subseteq D$ be the lunar disc diagram of minimal area satisfying (1) and (2). Then D' satisfies (3). To verify (4), let z_0 be an interior vertex of D' of valence 5 that is a neighbour of $m \geq 3$ vertices of $\partial D'$.

If $m = 5$, then by (3) z_0 is v-exposed. The same holds for $m = 4$, unless z_0 has exactly 2 neighbours on each γ_i', which are distinct from z, v, and consecutive by Corollary 2.6. We will discuss this possibility below, together with the case $m = 3$. Namely, if $m = 3$, then z_0 is u-exposed with $u \in \{z, v, v_1, v_2\}$, unless it has, say, two consecutive neighbours z_1, z_2 on γ_1' and a neighbour z_3 on γ_2', all of which are distinct from z, v. We can assume $d(z_1, v) = d(z_2, v) - 1$. Let $n = d(z_0, v)$. By the triangle inequality, we have $d(z_1, v) = n - 1$ or n, and $d(z_3, v) = n - 1, n$ or $n + 1$. In each of the cases we will prove that z_0 is u-exposed, with $u \in \{z, v, v_1, v_2\}$, or we will reach a contradiction by finding a properly contained lunar disc diagram $D'' \subsetneq D'$ between a vertex z' of D' and v satisfying (1) and (2) and hence contradicting the minimality hypothesis. Consider the *top* and *bottom* components obtained from D' by cutting along the path $z_1 z_0 z_3$ containing v, and z, respectively. Each of the two neighbours of z_0 distinct from z_1, z_2, z_3 is *top* (resp. *bottom*) if it lies in the top (resp. bottom) component.

Case 1: $d(z_1, v) = n - 1$.

a) $d(z_3, v) = n - 1$. In that case, if z_0 is not v-exposed, then $n \geq 3$ and we can take $z' = z_0$.

b) $d(z_3, v) = n$. If z_0 has exactly one bottom neighbour, then we can take z' to be that vertex. Note that the function from (2) is constant on all the vertices of D'' except for z' and z_0, since it is constant on z_1, z_2, z_3 and the top neighbour of z_0, which separate the remaining vertices of D'' from z' and z. If z_0 has two bottom neighbours, then we can take $z' = z_3$. If z_0 has no bottom neighbours, then z_2 is a neighbour of z_3, so they have a common neighbour $z' \neq z_0$ (which is distinct from z if z_0 is not z-exposed).

c) $d(z_3, v) = n + 1$. In that case, take $z' = z_3$.

Case 2: $d(z_1, v) = n$.

a) $d(z_3, v) = n - 1$. In that case, take $z' = z_2$.

b) $d(z_3, v) = n$. If z_0 has no bottom neighbours, then take $z' = z_2$ (which is distinct from z if z_0 is not z-exposed). If z_0 has exactly one bottom neighbour, then denote it z_4. If z_0 is not v-exposed, then $n \geq 2$. Then we can take as z' the common neighbour of z_2 and z_4 distinct from z_0 (which is distinct from z if z_0 is not z-exposed). Note that the function from (2) is constant on all the vertices of D'' except for z' and z_4, since it is constant on z_0, z_1, z_2, and z_3, which separate the remaining vertices of D'' from z' and z. If z_0 has two bottom neighbours, then this contradicts $d(z_0, v) = n$.

c) $d(z_3, v) = n + 1$. If z_0 has at least one bottom neighbour, we obtain a contradiction with $d(z_0, v) = n$. If z_0 has no bottom neighbours, then z_2 is a neighbour of z_3, and they have a common neighbour $z' \neq z_0$ (which is distinct from z if z_0 is not z-exposed).

□

Proof of Proposition 3.2. By Remark 3.4, and Lemma 3.3, we can assume that $D = D'$ and satisfies Lemma 3.3(3,4). For any interior vertex w of D', let $\kappa(w) = 6$ minus the valence of w. For w in $\partial D'$, let $\kappa(w) = 4$ minus the valence of w. By the combinatorial Gauss–Bonnet theorem (see e.g. [7, Thm 4.6]), the sum of all $\kappa(w)$ equals 6. For each interior vertex w of valence 5, let $\kappa'(w) = \kappa(w) - \frac{N}{3}$, where N is the number of the interior neighbours of w (all of which have valence ≥ 7). For each interior vertex w of valence ≥ 7, let $\kappa'(w) = \kappa(w) + \frac{N}{3}$, where N is the number of the interior neighbours of w of valence 5. We let $\kappa'(w) = \kappa(w)$ for the remaining w. Then the sum of all $\kappa'(w)$ equals 6 as well.

If there is a v-exposed vertex, we call it v'. If such a vertex does not exist, but there are v_i-exposed vertices, then we call them v'_i. If there is an x-exposed vertex, we call it x'.

Claim. *κ' is non-positive except possibly at*

- *x, v, where it is ≤ 2,*
- *v_i, where it is ≤ 1,*
- *u-exposed vertices, for $u \in \{x, v, v_1, v_2\}$, where it is ≤ 1.*

Indeed, if an interior vertex w of valence 5 is not u-exposed for $u \in \{x, v, v_1, v_2\}$, then by (4) we have $N \geq 3$, and so $\kappa'(w) = \kappa(w) - \frac{N}{3} \leq 1 - 1$. On the other hand, if an interior vertex w has valence 7, then we have $N \leq 3$ and so $\kappa'(w) = \kappa(w)+\frac{N}{3} \leq -1+1$, and if it has valence $k \geq 8$, then $N \leq \frac{k}{2}$ and so $\kappa'(w) = \kappa(w) + \frac{N}{3} \leq 6 - k + \frac{k}{6} = 6 - \frac{5k}{6} < 0$. This justifies the Claim.

To verify (i) or (ii) it suffices to check that

(i) $\kappa'(v) = 2$ and $\kappa(v_i) = 1$ for some i, or
(ii) $\kappa'(v) = \kappa'(v_1) = \kappa'(v_2) = 1$, and v' exists.

Note that if one of the v_i' exists, then $\kappa'(v_i) = 1$ and $\kappa'(v) \leq 0$. If both v_i' exist, then $\kappa'(v) \leq -1$.

Thus for the sum of all $\kappa'(w)$ to be equal to 6, the only remaining possibilities, up to a symmetry, are:

- $\kappa'(x) = \kappa'(v) = 2, \kappa'(x') = \kappa'(v') = 1, \kappa'(v_1) = \kappa'(v_2) = 0$,
- $\kappa'(x) = 2, \kappa'(v) = \kappa'(v_1) = \kappa'(v_2) = \kappa'(x') = 1$, and there is no v',
- $\kappa'(x) = 2, \kappa'(v) = \kappa'(v_1) = \kappa'(x') = \kappa'(v') = 1, \kappa'(v_2) = 0$, or
- $\kappa'(x) = 2, \kappa'(v_1) = \kappa'(v_2) = \kappa'(v_1') = \kappa'(v_2') = \kappa'(x') = 1, \kappa'(v) = -1$.

However, in all these cases, by (3), the vertex x' has at least one interior neighbour, which contradicts $\kappa'(x') = 1$. □

4. Contractibility

Lemma 4.1. *Suppose that K is a flag simplicial complex*

(1) of diameter ≤ 2,
(2) 5-large, and such that
(3) any induced 5-cycle is the boundary of a wheel of K.

Then K is contractible.

In the proof, we will use the following.

Lemma 4.2 ([5, Lem 8.11])**.** *Let K be as in Lemma 4.1. Then for any pair of simplices of K with vertex sets A_1, A_2, there is a vertex a of K that is a neighbour or equal to all of the elements of $A_1 \cup A_2$.*

Proof of Lemma 4.1. By Whitehead theorem, it suffices to show that any finite subcomplex K' of K is contained in a contractible subcomplex K'' of K. We consider all the subsets $V_0, \ldots, V_n$ of the vertex set of K' that span a simplex of K. Let M_0 be the simplex spanned on V_0. Using Lemma 4.2, we construct inductively simplices $M_1, \ldots, M_n$ so that $M_i \supseteq M_{i-1}$ and M_i contains a vertex a_i such that $V_i \cup \{a_i\}$ spans a simplex. Let K'' be the span of the union of K' and M_n. Note that K'' is flag and each maximal simplex of K'' intersects M_n. Then K'' is contractible (see e.g. [5, Lem 8.13]). □

An induced subcomplex C of a simplicial complex K is 3-*convex* if for every path abc with vertices a, c in C at distance 2 in K, we have that b also belongs to C.

Remark 4.3. Let K be as in Lemma 4.1. If C is a 3-convex subcomplex of K, then K also satisfies the hypotheses of Lemma 4.1.

4.1. **Downward links.** In the entire subsection, we assume that X is a simply connected, 7-located, locally 5-large simplicial complex.

We fix a basepoint vertex x of X. The *ball* $B_n(x)$ (resp. the *sphere* $S_n(x)$) is the subcomplex of X spanned by all the vertices at distance $\leq n$ (resp. $= n$) from x in X^1. Let $n > 0$ and let σ be a simplex contained in $S_n(x)$. The *link* of σ is the intersection of the links of all the vertices of σ, treated as subcomplexes of X. The intersection $K(v)$ of the link of σ with $S_{n-1}(x)$ is the *downward link* of v.

Our goal is to show that downward links satisfy the hypotheses of Lemma 4.1, and so they are contractible. To start with, let $\sigma = v$ be a vertex. Note that $K(v)$ satisfies Lemma 4.1(2) since X is locally 5-large.

From Proposition 3.2 it follows that $K(v)$ satisfies Lemma 4.1(1), and more generally:

Corollary 4.4. *Let v_1, v_2 be vertices of $K(v)$.*

(i) If v_1 and v_2 are neighbours, then $K(v_1)$ intersects $K(v_2)$.

(ii) If v_1 and v_2 are not neighbours, then they have a common neighbour v' in $K(v)$ such that there is an edge $v_1'v_2'$ with v_1' in $K(v_1v')$ and v_2' in $K(v_2v')$.

Proposition 4.5. *Any induced 5-cycle in $K(v)$ is the boundary of a wheel in $K(v)$.*

Proof. Let $\gamma = v_2w_2w_1v_1u$ be an induced 5-cycle in $K(v)$. Let v', v_1', v_2' be as in Corollary 4.4(ii). If v' is a neighbour of w_1 or w_2, then, since $K(v)$ is 5-large, we have that (v', γ) is the required wheel. Otherwise, $W_1 = (v, v_2w_2w_1v_1v')$ is a 5-wheel. Since $W_2 = (v', v_2v_2'v_1'v_1v)$ is also a 5-wheel, (W_1, W_2) is a $(5,5)$-dwheel, and so all the vertices of $W_1 \cup W_2$ have a common neighbour y of X. Since y is a neighbour of both v and v_1, we have that y is a vertex of $K(v)$. Since $K(v)$ is 5-large, considering the cycle v_1uv_2y we obtain that y is also a neighbour of u. Thus (y, γ) is the required wheel.

□

Corollary 4.6. *Each $K(v)$ satisfies the hypotheses of Lemma 4.1, and so it is contractible.*

Proposition 4.7. *Let $n > 0$, and let σ be a simplex of $S_n(x)$. Then $K(\sigma)$ is nonempty.*

Proof. Suppose first that σ is an edge of $S_n(x)$ with vertices v_1 and v_2. We may obtain a new complex X' by artificially adding to X a vertex v and a triangle vv_1v_2. This does not affect local 5-largeness or 7-location, so the proposition follows from Corollary 4.4(i) applied to v in X'.

Now suppose $\dim(\sigma) > 1$. We fix two distinct vertices v and y of σ. Let σ' be the subsimplex of σ spanned on all the vertices except for y, and let $e = vy$. By induction, we have vertices v_1 in $K(\sigma')$ and v_2 in $K(e)$. If neither v_1 nor v_2 lie in $K(\sigma)$, then there is $u \neq v$ in σ' that is not a neighbour of v_2, and y is not a neighbour of v_1. By the 5-largeness of the link of v, the vertex v_1 is not a neighbour of v_2. Let v', v_1', v_2' be the vertices from Corollary 4.4(ii). Note that if v' is a neighbour of y, then by the 5-largeness of the link of v it lies in $K(\sigma')$. Thus we can assume that v' is not a neighbour of y and so $W_1 = (v, v_2yuv_1v')$ is a 5-wheel. Since $W_2 = (v', v_2v_2'v_1'v_1v)$ is also a 5-wheel, (W_1, W_2) is a $(5,5)$-dwheel, and so all the vertices of $W_1 \cup W_2$ have a common neighbour z, which lies in $K(v)$. By the 5-largeness of the link of v, the vertex z lies in $K(\sigma')$.

□

Lemma 4.8. *Let $n > 0$, and let σ be a simplex of $S_n(x)$. Then for any vertex v of σ, the complex $K(\sigma)$ is a 3-convex subcomplex of $K(v)$.*

Before the proof, let us note that from Lemma 4.8, Remark 4.3, Lemma 4.1, Corollary 4.6, and Proposition 4.7, we deduce:

Corollary 4.9. *Each $K(\sigma)$ is contractible.*

Proof of Lemma 4.8. Let abc be a path in $K(v)$ with a, c in $K(\sigma)$ at distance 2 in $K(v)$. Let y be any vertex of σ distinct from v. Applying the 5-largeness of the link of v to the cycle $abcy$, we obtain that b is a neighbour of y. Since this holds for each y, we have that b belongs to $K(\sigma)$, as desired.

□

Proof of the Main Theorem. Let X be a 7-located locally 5-large simplicial complex. By passing to the universal cover of X, we can assume that X is simply connected. It suffices to prove that each $B_n(x)$ is contractible. To do this, it suffices to show that for each finite induced subcomplex A of $S_n(x)$, the span A_0 of $A \cup B_{n-1}(x)$ deformation retracts to $B_{n-1}(x)$. To this end, we order the simplices of A in the order of nonincreasing dimension $\sigma_1, \sigma_2, \ldots, \sigma_k$. Let A_i be the (not necessarily induced) subcomplex obtained from A_{i-1} by removing the open star of σ_i. Each such star is the join of σ_i with $K(\sigma_i)$, and so by Corollary 4.9, the complex A_{i-1} deformation retracts to A_i. Consequently, A_0 deformation retracts to $A_k = B_{n-1}(x)$. □

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